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Symmetry in Extended Phase Space for Singular Lagrangian with Subsidiary Constraints

Zi-ping Li[1,2] and Bi-chu Wu[2]

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A generalization of the Noether theorem to a singular nonholonomic system in the canonical formalism is given and its inverse theorem is presented. Based on the canonical action integral, a generalization of the Poincaré–Cartan integral invariant of a singular nonholonomic system is obtained. It is shown that this invariant is equivalent to the canonical equations of a singular constrained system. Some confusions in the literature are corrected. An example is given.

1. INTRODUCTION

The connection between continuous symmetry and conservation laws is usually referred to as the Noether theorem. The classical Noether theorem and its generalization (Li, 1981, 1984, 1985; Li and Li, 1990) are based on examination of the Lagrangian in configuration space and the corresponding transformation expressed in terms of Lagrange variables. For a system with a regular Lagrangian and a finite number of degrees of freedom, the invariance under the continuous transformation in terms of Hamiltonian variables was discussed by Djukic (1974). A system with a singular Lagrangian is subject to some inherent phase space constraint (Dirac, 1964; Sundermeyer, 1982) and is called a constrained Hamiltonian system. An example is a system in the gauge theories. Generalizations of the Noether theorem to a system with an ordinary singular Lagrangian and to a system with a singular higher-order Lagrangian in terms of canonical variables were discussed (Li and Li, 1991; Li, 1991).

The Poincaré–Cartan integral invariant plays a fundamental role in classical mechanics, hydromechanics, and field theories. The generalization

[1]CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China.
[2]Department of Applied Physics, Beijing Polytechnic University, Beijing 100022, China.

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of this invariant for a singular Lagrangian (Benavent and Gomis, 1979; Dominici and Gomis, 1980) and some applications (Sugano and Kamo, 1982; Sugano, 1982) have been given. The generalization of this invariant to a regular nonholonomic system was also given (Li and Li, 1990).

Dynamical systems (for example, classical mechanics and the mechanics of continuous media) are always subject to some subsidiary constraints; for some models of field theories the field variables are not independent, but there are some subsidiary constraint conditions among the field variables, such as the nonlinear σ-model and other models in field theories (Durand and Mandel, 1982). Here the symmetry properties in a constrained Hamiltonian system with subsidiary constraints (called a singular constrained system) are further investigated. For the sake of simplicity we consider a nonholonomic system with singular Lagrangian in classical mechanics. For extension to other singular constrained systems one can proceed in the same way (with possible modifications of some steps).

The paper is organized as follows. In Section 2, the generalized first Noether theorem (GFNT) in canonical formalism for a nonholonomic system with singular Lagrangian is deduced. In Section 3, an inverse theorem of GFNT is given. In Section 4, based on the canonical action, the Poincaré–Cartan invariant for singular Lagrangian with subsidiary nonlinear nonholonomic constraints is obtained. It is shown that this invariant is equivalent to the canonical equations of the singular constrained system, and some confusions in the literature are corrected. An example is given.

2. GFNT IN CANONICAL FORMALISM FOR SINGULAR CONSTRAINED SYSTEM

Consider a dynamical system with N degrees of freedom described by a singular Lagrangian $L(t, q, \dot{q})$ $(q = [q^1, q^2, \ldots, q^N])$. We introduce the canonical momenta $p_i = \partial L/\partial \dot{q}^i$ and Hamiltonian $H = p_i\dot{q}^i - L$ which may be formed by eliminating only $\dot{q}^i$ (the summation is taken over repeated indices). Take a system with singular Lagrangian whose Hessian matrix $(\partial^2 L/\partial \dot{q}^i\, \partial \dot{q}^j)$ is degenerate and suppose its rank to be $N - R$. Then the defining equations for the canonical momenta become noninvertible, and from the first $N - R$ equations only $N - R$ velocities $\dot{q}^\sigma$ can be solved as functions of t, q^i, $\dot{q}^a$, and p_σ

$$\dot{q}^\sigma = f^\sigma(t, q^i, \dot{q}^a, p_\sigma) \qquad (a = 1, 2, \ldots, R, \quad \sigma = R + 1, \ldots, N) \tag{1}$$

Substituting the $\dot{q}^\sigma$ in the last R defining equations for the canonical momenta yields R relations (primary constraints) between the canonical variables $(p = [p_1, p_2, \ldots, p_N])$:

$$\phi_a(t, q, p) = 0 \qquad (a = 1, 2, \ldots, R) \tag{2}$$

It was pointed out that Dirac's conjecture is invalid (Li and Li, 1991; Li, 1991), and the equations of motion of this system are given by (Sundermeyer, 1982)

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \mu^a \frac{\partial \phi_a}{\partial p_i} \qquad (i = 1, 2, \ldots, N) \tag{3a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i} - \mu^a \frac{\partial \phi_a}{\partial q^i} \tag{3b}$$

where $\mu^a(t)$ are Lagrange multipliers.

For a system whose motion is subject to subsidiary nonlinear nonholonomic constraints,

$$G_s(t, q, \dot{q}) = 0 \qquad (s = 1, 2, \ldots, M < N) \tag{4}$$

The equation of motion of this system is given by (Mei, 1985)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda^s \frac{\partial G_s}{\partial \dot{q}^i} \qquad (i = 1, 2, \ldots, N) \tag{5}$$

where $\lambda^s(t)$ are Lagrange multipliers.

Now we treat both subjects for systems that are described by a singular Lagrangian and, moreover, are submitted to some additional nonholonomic constraints. Let us suppose that we have taken into account the compatibility between the constraints arising from the singularity of the Lagrangian and the given nonholonomic constraints. For example, one can consider a simple case that one substitutes (1) into (4), which converts to the canonical constraints $G_s(t, q, p) = 0$ (see Section 5). From the variation of the Hamiltonian, one gets

$$\delta H = p_i \delta\dot{q}^i + \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i - \frac{\partial L}{\partial \dot{q}^i}\delta\dot{q}^i = \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \tag{6}$$

The Hamiltonian of the system depends only on time t and canonical variables for regular and singular Lagrangians (Nesterenko, 1989),

$$\delta H = \frac{\partial H}{\partial q^i}\delta q^i + \frac{\partial H}{\partial p_i}\delta p_i \tag{7}$$

Combining the expressions (6) and (7), one obtains

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\frac{\partial L}{\partial q^i} + \frac{\partial H}{\partial q^i}\right)\delta q^i = 0 \tag{8}$$

Substituting (5) into (8), one has

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} - \lambda^s \frac{\partial G_s}{\partial \dot{q}^i}\right)\delta q^i = 0 \tag{9}$$

For the singular Lagrangian, from (2) it follows that

$$\frac{\partial \phi_a}{\partial q^i}\,\delta q^i + \frac{\partial \phi_a}{\partial p_i}\,\delta p_i = 0 \tag{10}$$

Using the Lagrange multipliers $\mu^a(t)$ and combining the expressions (9) and (10), one obtains the equations of motion in the space of coordinates $(t, q, \dot{q}, p)$ for the singular Lagrangian with additional constraints (4):

$$\dot{q}^i = \frac{\partial H}{\partial p_i} + \mu^a \frac{\partial \phi_a}{\partial p_i} \tag{11a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} - \mu^a \frac{\partial \phi_a}{\partial q^i} + \lambda^s \frac{\partial G_s}{\partial \dot{q}^i} \tag{11b}$$

Let us consider an infinitesimal continuous r-parameter transformation of the time, generalized coordinates, and generalized momenta

$$\begin{cases} t \to \bar{t} = t + \delta t = t + \varepsilon_\sigma \tau^\sigma(t, q, p) \\ q^i(t) \to \bar{q}^i(\bar{t}) = q^i(t) + \Delta q^i(t) = q^i(t) + \varepsilon_\sigma \xi^{i\sigma}(t, q, p) \\ p_i(t) \to \bar{p}_i(\bar{t}) = p_i(t) + \Delta p_i(t) = p_i(t) + \varepsilon_\sigma \eta_i^\sigma(t, q, p) \end{cases} \tag{12}$$

Suppose that the canonical Lagrangian $L_p = p_i \dot{q}^i - H$ is gauge variant under the transformation (12), i.e., is invariant up to an exact differential term

$$\frac{d}{dt}(\delta\Omega) = \varepsilon_\sigma \frac{d\Omega^\sigma}{dt}$$

where ε_σ $(\sigma = 1, 2, \ldots, r)$ are parameters, $\Omega^\sigma = \Omega^\sigma(t, q, p)$. The variation of the canonical action

$$I_p = \int_{t_1}^{t_2} L_p\, dt = \int_{t_1}^{t_2} [p_i \dot{q}^i - H(t, q, p)]\, dt \tag{13}$$

is given by

$$\Delta I_p = \int_{\bar{t}_1}^{\bar{t}_2} \bar{L}_p(\bar{t}, \bar{q}, \bar{p})\, d\bar{t} - \int_{t_1}^{t_2} L_p(t, q, p)\, dt = \int_{t_1}^{t_2} \frac{d}{dt}(\delta\Omega)\, dt \tag{14}$$

From (12) and (14) one has

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)\delta p_i - \left(\dot{p}^i + \frac{\partial H}{\partial q^i}\right)\delta q^i + \frac{d}{dt}(p_i \delta q^i + L_p\, \Delta t - \delta\Omega) = 0 \tag{15}$$

where δp_i and δq^i are simultaneous variations of p_i and q^i,

$$\delta q^i = \Delta q^i - \dot{q}^i\, \Delta t, \qquad \delta p_i = \Delta p_i - \dot{p}_i\, \Delta t \tag{16}$$

Under the transformation (12) suppose that the change of ϕ_a is given by $\Delta\phi_a = \varepsilon_\sigma K_a^\sigma$; then one has

$$\delta\phi_a = \frac{\partial\phi_a}{\partial q^i}\delta q^i + \frac{\partial\phi_a}{\partial p_i}\delta p_i = \varepsilon_\sigma F_a^\sigma = F_a \tag{17}$$

where

$$F_a^\sigma = K_a^\sigma - \frac{\partial\phi_a}{\partial t}\tau^\sigma - \frac{\partial\phi_a}{\partial q^i}\dot{q}^i\tau^\sigma - \frac{\partial\phi_a}{\partial p_i}\dot{p}_i\tau^\sigma \tag{18}$$

If the simultaneous variation δq^i determined by the transformation (12) satisfies the same conditions as the virtual displacement imposed by constraints (4) (i.e., a nonholonomic system of Chetaev type), then

$$\frac{\partial G_s}{\partial \dot{q}^i}\delta q^i = 0 \tag{19}$$

Using a set of Lagrange multipliers $\lambda^s(t)$ and $\mu^a(t)$, combining the expressions (15), (17), and (19), one obtains

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i} - \mu^a\frac{\partial\phi_a}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} + \mu^a\frac{\partial\phi_a}{\partial q^i} - \lambda^s\frac{\partial G_s}{\partial\dot{q}^i}\right)\delta q^i$$
$$+ \frac{d}{dt}(p_i\delta q^i + L_p\,\Delta t - \delta\Omega) = \lambda^a F_a \tag{20}$$

Along the trajectory of motion of a singular constrained system, from (11) one obtains

$$\frac{d}{dt}[p_i(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) + L_p\tau^\sigma - \Omega^\sigma] = \lambda^a F_a^\sigma \tag{21}$$

Therefore, we have the following GFNT in the canonical formalism for a singular constrained system: If, under the transformation (12), the canonical Lagrangian L_p is invariant up to an exact differential term and the generators τ^σ, $\xi^{i\sigma}$, and η_i^σ of the transformation (12) satisfy

$$\frac{\partial G_s}{\partial\dot{q}^i}(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) = 0 \qquad (s = 1, 2, \ldots, M, \quad \sigma = 1, 2, \ldots, r) \tag{22}$$

$$\frac{\partial\phi_a}{\partial q^i}(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) + \frac{\partial\phi_a}{\partial p_i}(\eta_i^\sigma - \dot{p}_i\tau^\sigma) = 0 \quad (a = 1, 2, \ldots, R, \quad \sigma = 1, 2, \ldots, r) \tag{23}$$

then the expression

$$p_i\xi^{i\sigma} - H\tau^\sigma - \Omega^\sigma = \text{const} \qquad (\sigma = 1, 2, \ldots, r) \tag{24}$$

represents the constants of the motion. This theorem is a generalization of the previous result (Li and Li, 1991).

3. THE INVERSE THEOREM OF THE GFNT

We now find the conditions under which the inversion of the GFNT is possible in the canonical formalism for a singular constrained system. Suppose that we know r independent conservative quantities in phase space for a singular nonholonomic system which is given by

$$D^{\sigma}(t, q, p) = C^{\sigma} = \text{const} \qquad (\sigma = 1, 2, \ldots, r) \tag{25}$$

Using these conservative quantities, we are going to find the corresponding transformation (12) under which the variation of the canonical action (13) is given by (14). For a singular nonholonomic system, any dynamical trajectory of the motion satisfying equations (11), it follows that

$$\left(\dot{q}^i - \frac{\partial H}{\partial p_i} - \mu^a \frac{\partial \phi_a}{\partial p_i}\right)\bar{\eta}_i^{\sigma} - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} + \mu^a \frac{\partial \phi_a}{\partial q^i} - \lambda^s \frac{\partial G_s}{\partial \dot{q}^i}\right)\bar{\xi}^{i\sigma} = 0 \tag{26}$$

where $\varepsilon_{\sigma}\bar{\eta}_i^{\sigma} = \delta p_i$, $\varepsilon_{\sigma}\bar{\xi}^{i\sigma} = \delta q^i$. Combining expression (26) and the derivative with respect to time t of expression (25), one has

$$\begin{aligned}\frac{\partial D^{\sigma}}{\partial t} + \frac{\partial D^{\sigma}}{\partial q^i}\dot{q}^i + \frac{\partial D^{\sigma}}{\partial p_i}\dot{p}_i + \left(\dot{q}^i - \frac{\partial H}{\partial p_i} - \mu^a \frac{\partial \phi_a}{\partial p_i}\right)\bar{\eta}_i^{\sigma} \\ - \left(\dot{p}_i + \frac{\partial H}{\partial q^i} + \mu^a \frac{\partial \phi_a}{\partial q^i} - \lambda^s \frac{\partial G_s}{\partial \dot{q}^i}\right)\bar{\xi}^{i\sigma} = 0\end{aligned} \tag{27}$$

These relations are to be fulfilled for any dynamical trajectory of the motion; this leads to terms containing the time derivatives of p_i canceling each other; thus

$$\bar{\xi}^{i\sigma} = \partial D^{\sigma}/\partial p_i \tag{28}$$

Now let

$$D^{\sigma} = p_i\bar{\xi}^{i\sigma} + L_p\tau^{\sigma} - \Omega^{\sigma} \tag{29}$$

Hence

$$\tau^{\sigma} = L_p^{-1}(D^{\sigma} - p_i\bar{\xi}^{i\sigma} + \Omega^{\sigma}) \tag{30}$$

From (28) and (30) one can find $\xi^{i\sigma}$.

In addition, the quantities η_i^{σ} appear in the transformation laws for the generalized momenta in (12); at first it may appear that these quantities may be arbitrary, but if we recall the definition of canonical momenta, generalized momenta are known functions of time, generalized coordinates, and generalized velocities,

$$\bar{p}_i = \bar{p}_i(\bar{t}, \bar{q}, \dot{\bar{q}}) \qquad \text{and} \qquad \dot{\bar{q}} = \dot{q} + \Delta\dot{q}$$

from (12) one has (Djukic, 1974)

$$\Delta\dot{q}^i = \frac{d\bar{q}^i}{d\bar{t}} - \frac{dq^i}{dt} = \varepsilon_\sigma\left(\frac{d\xi^{i\sigma}}{dt} - \dot{q}^i\frac{d\tau^\sigma}{dt}\right) \tag{31}$$

Hence the transformed generalized momenta $\bar{p}_i$ are completely determined in terms of τ^σ, $\xi^{i\sigma}$, and untransformed canonical variables; the corresponding quantities η_i^σ in transformed generalized momenta can be obtained as functions of τ^σ, $\xi^{i\sigma}$, and untransformed canonical variables. Therefore, we have found the generators $\tau^\sigma, \xi^{i\sigma}, \eta_i^\sigma$ of the transformation (12). If these generators satisfy the conditions

$$\frac{\partial G_d}{\partial \dot{q}^i}(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) = 0 \tag{32}$$

$$\frac{\partial \phi_a}{\partial q^i}(\xi^{i\sigma} - \dot{q}^i\tau^\sigma) + \frac{\partial \phi_a}{\partial p_i}(\eta_i^\sigma - \dot{p}_i\tau^\sigma) = 0 \tag{33}$$

then the transformation (12) is generated by these generators τ^σ, $\xi^{i\sigma}$, and η_i^σ under which the variation of the canonical action (13) satisfies (14). In fact, substituting the generators $\tau^\sigma, \xi^{i\sigma}, \eta_i^\sigma$ and (32) and (33) into the expression (27), one gets

$$\frac{d}{dt}(p_i\bar{\xi}^{i\sigma} + L_p\tau^\sigma) + \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right)\bar{\eta}_i^\sigma - \left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right)\bar{\xi}^{i\sigma} = \dot{\Omega}^\sigma \tag{34}$$

Multiplying expression (34) by ε_σ, taking the summation for dummy upper and lower indices σ, and integrating the result, one obtains expression (14). Therefore, we obtain the inverse theorem of the GFNT in the canonical formalism for a singular constrained system: To any r-independent constant of motion $D^\sigma(t, q, p)$ for a system with a singular Lagrangian and constraints (4) there corresponds an infinitesimal transformation (12) generated by the above generators τ^σ, $\xi^{i\sigma}$, and η_i^σ, and so long as the conditions (32) and (33) are satisfied, this transformation induces a variation (14) on the canonical action of the system.

4. POINCARÉ–CARTAN INTEGRAL INVARIANT FOR SINGULAR CONSTRAINED SYSTEM

Let us consider a dynamical system with constraints (4) described by a singular Lagrangian $L(t, q, \dot{q})$. This system also has canonical constraints (2). Suppose that the equations of motion of this singular constrained system are given by (11). Now let us consider the transformation

$$\begin{cases} t \to \bar{t} = t + \Delta t(\alpha) \\ q^i(t) \to \bar{q}^i(\bar{t}) = q^i(t) + \Delta q^i(t, \alpha) \\ p_i(t) \to \bar{p}_i(\bar{t}) = p_i(t) + \Delta p_i(t, \alpha) \end{cases} \tag{35}$$

where α is a parameter which satisfies

$$q^i(t, 0) = q^i(t), \qquad p_i(t, 0) = p_i(t) \tag{36}$$

Under the transformation (35) the variation of the canonical action (13) is given by

$$\begin{aligned} \Delta I_p &= I_p'(\alpha)\, \Delta\alpha \\ &= \int_{t_1}^{t_2} \left[\left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) \delta p_i - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q^i + \frac{d}{dt}(p_i\, \Delta q^i - H\, \Delta t) \right] dt \end{aligned} \tag{37}$$

where δp_i and δq^i are simultaneous variations of p_i and q^i which are given by (16).

Let the simultaneous variations δq^i determined by the transformation (35) satisfy the virtual displacement conditions (19) imposed by constraints (4), and suppose that the simultaneous variations δp_i and δq^i satisfy the condition:

$$\frac{\partial \phi_a}{\partial q^i} \delta q^i + \frac{\partial \phi_a}{\partial p_i} \delta p_i = 0 \tag{38}$$

Introducing the Lagrange multipliers $\lambda^s(t)$ and $\mu^a(t)$ and combining with the expressions (19), (37), and (38), one gets

$$\begin{aligned} \Delta I_p = I_p'(\alpha)\, \Delta\alpha = \int_{t_1}^{t_2} \Bigg\{ & \left[\dot{q}^i - \frac{\partial H}{\partial p_i} - \mu^a \frac{\partial \phi_a}{\partial p_i} \right] \delta p_i \\ & - \left[\dot{p}_i + \frac{\partial H}{\partial q^i} + \mu^a \frac{\partial \phi_a}{\partial q^i} - \lambda \frac{\partial G_s}{\partial \dot{q}^i} \right] \delta q^i + \frac{d}{dt}[p_i\, \Delta q^i - H\, \Delta t] \Bigg\} dt \end{aligned} \tag{39}$$

Along the dynamical trajectory of motion of the singular constrained system, using equations (11), one obtains

$$\Delta I_p = I_p'(\alpha)\, \Delta\alpha = [p_i\, \Delta q^i - H\, \Delta t]_1^2 \tag{40}$$

Let C_1 be any simple closed curve encircling the tube of dynamical trajectories in extended phase space, i.e., through any point on C_1 there is a dynamical trajectory of the motion. The equation of this closed curve C_1 is given by

$$t = t_{(1)}(\alpha), \qquad q^i = q^i_{(1)}(\alpha), \qquad p_i = p_i^{(1)}(\alpha) \tag{41}$$

where $\alpha = 0$ and $\alpha = l$ is the same point on C_1. The dynamical trajectories through every point on C_1 form a tube of trajectories

$$q^i = q^i(t, \alpha), \qquad p_i = p_i(t, \alpha) \tag{42}$$

where $q^i(t, 0) = q^i(t, l), p_i(t, 0) = p_i(t, l)$. Choose another closed curve C_2 on this tube of trajectories such that it intersects the generatrix of the tube once. Suppose the equation of C_2 is given by

$$t = t_{(2)}(\alpha), \qquad q^i = q^i_{(2)}(\alpha), \qquad p_i = p_i^{(2)}(\alpha) \tag{43}$$

where $\alpha = 0$ and $\alpha = l$ is also the same point on C_2. Along the curves C_1 and C_2 take the integral of the expression (40) with respect to α, which gives the same result, respectively, i.e.,

$$J = \oint_C [p_i\, \Delta q^i - H\, \Delta t] = inv \tag{44}$$

where C is any simple closed curve encircling the tube of dynamical trajectories. This integral (44) calculated along an arbitrary closed contour lying on the hypersurface of extended phase space (t, q^i, p_i), defined by constraint equations (2) and (4), is invariant under an arbitrary displacement (with deformation) of the contour along any tube of dynamical trajectories. J is called the Poincaré–Cartan integral invariant for the singular nonholonomic constrained system.

Conversely, let us suppose we have a singular constrained system, with constraint equations given by (2) and (4), whose dynamical trajectories satisfy a system of differential equations

$$\dot{q}^i = f^i(t, q, p, \lambda, \mu), \qquad \dot{p}_i = g_i(t, q, p, \lambda, \mu) \qquad (i = 1, 2, \ldots, n) \tag{45}$$

where f^i, g_i depend on some arbitrary functions. Then we can show that the sufficient condition for equations (45) to be the canonical equation (11) of a singular constrained system is that the Poincaré–Cartan integral (44) be invariant.

In fact, following Gantmacher (1979), we introduce an auxiliary variable, using the Poincaré–Cartan integral invariant (44), and obtain (Dominici and Gomis, 1980)

$$\left(g_i + \frac{\partial H}{\partial q^i}\right) \Delta q^i + \left(-f^i + \frac{\partial H}{\partial p_i}\right) \Delta p_i + \left(-\frac{dH}{dt} + \frac{\partial H}{\partial t}\right) \Delta t = 0 \tag{46}$$

The variations Δq^i and Δp_i are not independent; one has

$$\frac{\partial G_s}{\partial \dot{q}^i} (\Delta q^i - \dot{q}^i\, \Delta t) = 0 \tag{47}$$

$$\frac{\partial \phi_a}{\partial q_i} (\Delta q^i - \dot{q}^i\, \Delta t) + \frac{\partial \phi_a}{\partial p_i} (\Delta p_i - \dot{p}_i\, \Delta t) = 0 \tag{48}$$

Introducing a set of Lagrange multipliers $\lambda^s(t)$ and $\mu^\sigma(t)$, combining the expressions (46)–(48), one obtains

$$\dot{q}^i = f^i = \frac{\partial H}{\partial p_i} + \mu^a \frac{\partial \phi_a}{\partial p_i} \tag{49}$$

$$\dot{p}_i = g_i = -\frac{\partial H}{\partial q^i} - \mu^a \frac{\partial \phi_a}{\partial q^i} + \lambda^s \frac{\partial G_s}{\partial \dot{q}^i} \tag{50}$$

$$-\frac{dH}{dt} + \frac{\partial H}{\partial t} + \lambda^s \frac{\partial G_s}{\partial \dot{q}^i} \dot{q}^i - \mu^a \left(\frac{\partial \phi_a}{\partial q^i} \dot{q}^i + \frac{\partial \phi_a}{\partial p_i} \dot{p}_i \right) = 0 \tag{51}$$

Equations (49) and (50) are just the canonical equations of the singular nonholonomic system.

There is some confusion in the literature (Benavent and Gomis, 1979; Dominici and Gomis, 1980) regarding the total variation and simultaneous variation: It was required that the constraint conditions (2) are invariant under the total variation of canonical variables q^i and p_i, and in that case there are no constraint conditions (4). Thus, one has

$$\frac{\partial \phi_a}{\partial q^i} \Delta q^i + \frac{\partial \phi_a}{\partial p_i} \Delta p_i = 0 \tag{52}$$

According to the consistency condition of constraint,

$$\begin{aligned} \dot{\phi}_a &= \frac{\partial \phi_a}{\partial t} + \frac{\partial \phi_a}{\partial q^i} \dot{q}^i + \frac{\partial \phi_a}{\partial p_i} \dot{p}_i \\ &= \frac{\partial \phi_a}{\partial t} + \frac{\partial \phi_a}{\partial q^i} \left(\frac{\partial H}{\partial p_i} + \mu^b \frac{\partial \phi_b}{\partial p_i} \right) + \frac{\partial \phi_a}{\partial p_i} \left(-\frac{\partial H}{\partial q^i} - \mu^b \frac{\partial \phi_b}{\partial q^i} \right) \\ &= \frac{\partial \phi_a}{\partial t} + \{\phi_a, H + \mu^b \phi_b\} = 0 \end{aligned} \tag{53}$$

where $\{\cdot, \cdot\}$ denote the Poisson bracket. From (16), (35), (52), and (53), one can obtain

$$[p_i \, \Delta q^i - H \, \Delta t]_1^2 = \int_{t_1}^{t_2} \lambda^a \frac{\partial \phi_a}{\partial t} \Delta t \, dt \tag{54}$$

In general, from (54) one cannot obtain the Poincaré–Cartan integral invariant (44), unless ϕ_a does not depend on time explicitly or Δt equals zero. Thus, the condition (38) is necessary for deriving the invariant (44).

In addition, if the condition (52) holds, from (46) and (52) one can conclude that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{55}$$

This equality (55) holds for a regular Lagrangian (Gantmacher, 1979), but for the singular Lagrangian in that case (Benavent and Gomis, 1979; Dominici and Gomis, 1980), from (53) and

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\{H, H+\mu^b\phi_b\}=\frac{\partial H}{\partial t}+\mu^b\{H, \phi_b\} \tag{56}$$

one can obtain

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}+\mu^a\frac{\partial \phi_a}{\partial t} \tag{57}$$

Hence, the equality (55) does not hold for a singular Lagrangian unless ϕ_a does not depend on time t explicitly. Therefore, in general one must be careful to distinguish the total variation and simultaneous variation of canonical variables and use (38) instead of (52) to derive the Poincaré–Cartan integral invariant (44).

5. AN EXAMPLE

Now we present an example for the inverse theorem of the GFNT. Let us consider the mechanical system in Euclidean space with coordinates $x(t), y(t), z(t)$ whose Lagrangian is given by (Galvão and Boechat, 1990)

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)+\tfrac{1}{2}z^2(x^2+y^2)-z(x\dot{y}-y\dot{x})-\tfrac{1}{2}(x^2+y^2) \tag{58}$$

Suppose that this system is subject to a subsidiary nonlinear nonholonomic constraint which is given by

$$G=a^2\dot{x}^2-\dot{x}\dot{y}-\dot{y}^2=0 \qquad (a=\text{const}) \tag{59}$$

Because the constraint (59) is a homogeneous function with respect to $\dot{x}$ and $\dot{y}$ and $\partial L/\partial t=0$, it is easy to find that the energy of this nonholonomic system is conservative (Li, 1988).

It is clear that this Lagrangian (58) is singular since z is not a dynamical variable. The canonical moments are given by

$$p_x=\dot{x}+zy, \qquad p_y=\dot{y}-zx, \qquad p_z=0 \tag{60}$$

Hence

$$\phi=p_z=0 \tag{61}$$

is a constraint. The canonical Hamiltonian is given by

$$H=\tfrac{1}{2}(p_x^2+p_y^2)+z(xp_y-yp_x)+\tfrac{1}{2}(x^2+y^2) \tag{62}$$

The generalized energy conservation of this system implies that

$$D(t, q, p) = \tfrac{1}{2}(p_x^2 + p_y^2) + z(xp_y - yp_x) + \tfrac{1}{2}(x^2 + y^2) = \text{const} \tag{63}$$

From expression (28), one obtains

$$\bar{\xi}_1 = \bar{\xi}_x = p_x - zy, \qquad \bar{\xi}_2 = \bar{\xi}_y = p_y + zx, \qquad \bar{\xi}_3 = \bar{\xi}_z = 0 \tag{64}$$

and using relation (30), one finds

$$\tau = L_p^{-1}[-\tfrac{1}{2}(p_x^2 + p_y^2) - yzp_x + xzp_y - \Omega] \tag{65}$$

Let us choose

$$\Omega = -\tfrac{1}{2}(p_x^2 + p_y^2) - (yzp_x - xzp_y) \tag{66}$$

Then $\tau = 0$, and consequently one finds the generators $\tau = 0$, $\xi_1 = p_x - zy$, $\xi_2 = p_y + zx$, and $\xi_3 = 0$. Using expression (60), one obtains $\eta_1 = \dot{\xi}_1 + y\xi_2$, $\eta_2 = \dot{\xi}_2 - z\xi_1$, and $\eta_3 = 0$.

It is easy to verify that $\xi_1 = p_x - zy = \dot{x}$, $\xi_2 = p_y + zx = \dot{y}$, and $\xi_3 = 0$, and η_1, η_2, η_3 satisfy conditions (32) and (33) for constraints (59) and (61), respectively. Finally, we find the following infinitesimal transformation in connection with the inverse theorem of the GFNT in canonical form:

$$\begin{cases} \bar{t} = t, & \bar{p}_x = p_x + \varepsilon(\dot{\xi}_1 + y\xi_2) \\ \bar{x} = x + \varepsilon\xi_1, & \bar{p}_y = p_y + \varepsilon(\dot{\xi}_2 - z\xi_1) \\ \bar{y} = y + \varepsilon\xi_2. & \bar{p}_z = p_z \\ \bar{z} = z, & \end{cases} \tag{67}$$

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